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Generating mapping class groups with elements of fixed finite order

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A R T I C L E I N F O

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We show that for $k \geq 6$ and g sufficiently large, the mapping class group of a surface of genus g can be generated by three elements of order k. We also show that this can be done with four elements of order 5 when g is at least 8.

1. Introduction

Let S_g be a closed, connected, and orientable surface of genus g. The mapping class group $\mathrm{Mod}(S_g)$ is the group of homotopy classes of orientation-preserving homeomorphisms of S_g. In this paper, we construct small generating sets for $\mathrm{Mod}(S_g)$ where all of the generators have the same finite order.

Theorem 1.1. *Let $k \geq 6$ and $g \geq (k-1)^2 + 1$. Then $\mathrm{Mod}(S_g)$ is generated by three elements of order k. Also, $\mathrm{Mod}(S_g)$ is generated by four elements of order 5 when $g \geq 8$.*

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Theorem 1.1 follows from a stronger but more technical result that we prove as Theorem 4.1. Our generating sets for $\mathrm{Mod}(S_g)$ are constructed explicitly. In addition, the elements in any particular generating set are all conjugate to each other. Of course, attempting to construct generating sets consisting of elements of a fixed order k only makes sense if $\mathrm{Mod}(S_g)$ contains elements of order k in the first place. A construction of Tucker [19] guarantees an element of any fixed order k in $\mathrm{Mod}(S_g)$ whenever g is sufficiently large, as described in Section 2.

Later in the introduction we describe prior work by several authors on generating $\mathrm{Mod}(S_g)$ with elements of fixed finite orders 2, 3, 4, and 6. In each case, the authors show that the number of generators required is independent of g. Set alongside this prior work, a new phenomenon that emerges in our results is that the sizes of our generating sets for $\mathrm{Mod}(S_g)$ are not only independent of the genus of the surface, but they are also independent of the order of the elements.

Our generators are all finite-order elements that can be realized by rotations of S_g embedded in $\mathbb{R}^3$. There are values of g and k where there exist elements of order k in $\mathrm{Mod}(S_g)$, but where these cannot be realized as rotations of S_g embedded in $\mathbb{R}^3$. For instance, there are elements of order 7 in $\mathrm{Mod}(S_3)$ that cannot be realized in this way.

Problem 1.2. *Extend Theorem 4.1 to cases where elements of order k exist in $\mathrm{Mod}(S_g)$ but cannot be realized as rotations of S_g embedded in $\mathbb{R}^3$.*

We can also seek smaller generating sets for $\mathrm{Mod}(S_g)$ consisting of elements of order k. We note that any such sharpening of Theorem 4.1 would seem to demand a new approach. Our proofs hinge on applications of the lantern relation, and a lantern has only a limited number of symmetries.

Problem 1.3. *For fixed $k \geq 3$ and any $g \geq 3$ where elements of order k exist, can $\mathrm{Mod}(S_g)$ be generated by two elements of order k? What about three elements for orders 4 and 5?*

Background and prior results. The most commonly-used generating sets for $\mathrm{Mod}(S_g)$ consist of Dehn twists, which have infinite order. Dehn [4] showed that $2g(g-1)$ Dehn twists generate $\mathrm{Mod}(S_g)$, and Lickorish [14] showed that $3g+1$ Dehn twists suffice. Humphries [9] showed that only $2g+1$ Dehn twists are needed, and he also showed that no smaller set of Dehn twists can generate $\mathrm{Mod}(S_g)$. The curves for these Dehn twists are depicted in Fig. 1.1.

There have also been many investigations into constructing generating sets for $\mathrm{Mod}(S_g)$ that include or even consist entirely of periodic elements. For instance, Maclachlan [16] showed that $\mathrm{Mod}(S_g)$ is normally generated by a set of two periodic elements that have orders $2g+2$ and $4g+2$, and McCarthy and Papadopoulos [17] showed that $\mathrm{Mod}(S_g)$ is normally generated by a single involution (element of order 2) for $g \geq 3$. Korkmaz [12] showed that $\mathrm{Mod}(S_g)$ is generated by two elements of order $4g+2$ for $g \geq 3$.

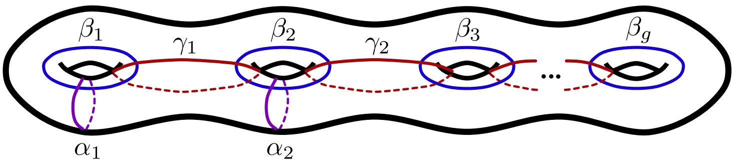

Fig. 1.1. The $2g+1$ Humphries curves in S_g.

Luo [15] explicitly constructed a finite generating set for $\mathrm{Mod}(S_g)$ consisting of $6(2g+1)$ involutions, given that $g \geq 3$. Luo asked whether there exists a universal upper bound (that is, independent of g) for the number of involutions required to generate $\mathrm{Mod}(S_g)$. Brendle and Farb [2] showed that six involutions suffice to generate $\mathrm{Mod}(S_g)$, again for $g \geq 3$. Kassabov [10] sharpened this result by showing that only five involutions are needed for $g \geq 5$ and only four are needed for $g \geq 7$. Monden [18] showed that $\mathrm{Mod}(S_g)$ can be generated by three elements of order 3 and by four elements of order 4, each for $g \geq 3$. Recently Yoshihara [20] has shown that $\mathrm{Mod}(S_g)$ can be generated by three elements of order 6 when $g \geq 10$ and by four elements of order 6 when $g \geq 5$.

Much work has been done to establish when elements of a particular finite order exist in $\mathrm{Mod}(S_g)$. In his paper, Monden noted that for all $g \geq 1$, $\mathrm{Mod}(S_g)$ contains elements of orders 2, 3, and 4. Aside from order 6, elements of larger orders do not always exist in $\mathrm{Mod}(S_g)$. For example, $\mathrm{Mod}(S_3)$ contains no element of order 5 and $\mathrm{Mod}(S_4)$ contains no element of order 7.

Determining the orders of the periodic elements in $\mathrm{Mod}(S_g)$ for any particular g is a solved problem, at least implicitly. In fact, this is even true for determining the conjugacy classes of periodic elements in $\mathrm{Mod}(S_g)$. Ashikaga and Ishizaka [1] listed necessary and sufficient criteria for determining the conjugacy classes in $\mathrm{Mod}(S_g)$ for any particular g. The criteria are number theoretic and consist of the Riemann–Hurwitz formula, an upper bound of $4g+2$ on the order of elements due to Wiman, an integer-sum condition on the valencies of the ramification points due to Nielsen, and several conditions on the least common multiple of the ramification indices that are due to Harvey.

Ashikaga and Ishizaka also gave lists of the conjugacy classes of periodic elements in $\mathrm{Mod}(S_1)$, $\mathrm{Mod}(S_2)$, and $\mathrm{Mod}(S_3)$. Hirose [8] gave a list of the conjugacy classes of periodic elements in $\mathrm{Mod}(S_4)$. Broughton [3] listed criteria for determining actions of finite groups on S_g, and hence for determining conjugacy classes of finite subgroups of $\mathrm{Mod}(S_g)$. Broughton also gave a complete classification of actions of finite groups on S_2 and S_3. Kirmura [11] gave a complete classification for S_4.

Several results have been proved about guaranteeing the existence of elements of order k in $\mathrm{Mod}(S_g)$ for sufficiently large g. Harvey [7] showed that $\mathrm{Mod}(S_g)$ contains an element of order k whenever $g \geq (k^2-1)/2$. Glover and Mislin [6] showed that $\mathrm{Mod}(S_g)$ contains an element of order k whenever $g > (2k)^2$. A fundamental result in this direction was shown by Kulkarni [13]: for any finite group G, the g for which G acts faithfully on S_g all fall in some infinite arithmetic progression; and further, all but finitely many values in the arithmetic progression are admissible g.

Tucker [19] gave necessary and sufficient conditions for the existence of an element of order k in $\mathrm{Mod}(S_g)$ that can be realized as a rotation of S_g embedded in $\mathbb{R}^3$. Using this characterization, Tucker showed that for any k and for sufficiently large g, $\mathrm{Mod}(S_g)$ contains an element of order k that is realizable by a rotation of S_g embedded in $\mathbb{R}^3$. We give a proof of this fact in Lemma 2.1.

Outline of the paper. In Section 2 we show how to construct elements of order k in $\mathrm{Mod}(S_g)$ for sufficiently large values of g. From here, our strategy for constructing generating sets for $\mathrm{Mod}(S_g)$ unfolds as follows. As mentioned before, Humphries showed that the Dehn twists about the $2g+1$ curves in Fig. 1.1 generate $\mathrm{Mod}(S_g)$. To show that a collection of elements generates $\mathrm{Mod}(S_g)$, then, it suffices to show two things: that a Dehn twist about some Humphries curve can be written as a product in these elements, and that all $2g+1$ Humphries curves are in the same orbit under the subgroup generated by these elements. If these two things hold, it follows that the Dehn twist about each of the Humphries curves may be written as a product in the elements, and therefore the elements generate $\mathrm{Mod}(S_g)$. We show how to write a Dehn twist as a product in elements of order k in Section 3. In Sections 4 and 5 we prove our main technical result by constructing generating sets for $\mathrm{Mod}(S_g)$ that are comprised of elements of fixed finite order.

2. Constructing elements of a given order in $\mathrm{Mod}(S_g)$

In this section we construct elements of order k in $\mathrm{Mod}(S_g)$ whenever g is sufficiently large. We will use elements that are conjugate to these elements when we build our generating sets.

The following result gives sufficient conditions for the existence of elements of order k in $\mathrm{Mod}(S_g)$ that can be realized by a rotation of S_g embedded in $\mathbb{R}^3$. The result was proved by Tucker [19], who additionally showed that these sufficient conditions are in fact necessary. We include a proof of the result in order to establish conventions about the geometric realizations of the elements it guarantees, as these geometric realizations will be important in proving our main result.

Lemma 2.1. *Let $k \geq 2$. Then $\mathrm{Mod}(S_g)$ contains an element of order k that can be realized as a rotation of S_g embedded in $\mathbb{R}^3$ whenever $g > 0$ can be written as $ak + b(k-1)$ with $a, b \in \mathbb{Z}_{\geq 0}$ or as $ak + 1$ with $a \in \mathbb{Z}_{>0}$.*

Proof. In Fig. 2.1 we depict two ways of embedding a surface in $\mathbb{R}^3$ so that it has k-fold rotational symmetry. First, we can embed a surface of genus k in $\mathbb{R}^3$ so that it has a rotational symmetry of order k by evenly spacing k handles about a central sphere. We can also embed a surface of genus $k-1$ in $\mathbb{R}^3$ so that it has a rotational symmetry of order k, as follows. Arrange two spheres along an axis of rotation and remove k disks from each sphere, evenly spaced along the equator of each. Then connect pairs of boundary

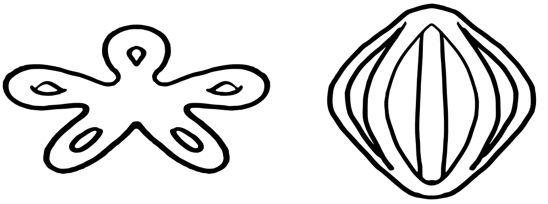

Fig. 2.1. Embeddings of S_5 and S_4 with rotational symmetry of order 5.

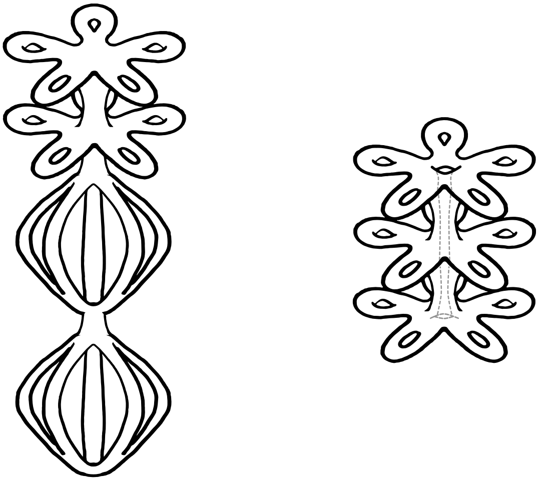

Fig. 2.2. Embeddings of S_{18} and S_{16} with rotational symmetry of order 5.

components, one from each sphere, with a cylinder. This can be done symmetrically so that a rotation by $2\pi/k$ permutes the cylinders cyclically.

We can use these two types of embeddings to construct embeddings of surfaces of higher genus that also have rotational symmetry of order k. Whenever $g = ak + b(k-1)$, we can construct an embedding of S_g in $\mathbb{R}^3$ by taking a connected sum of surfaces of genus k and $k-1$ along their axis of rotational symmetry. See the left of Fig. 2.2 for an example. Rotating a surface embedded in this way by $2\pi/k$ produces an element of $\mathrm{Mod}(S_g)$ of order k for any genus $g = ak + b(k-1)$. That an element so formed does not have order less than k can be seen by the element's action on homology.

In order to produce elements of order k in the case where $g = ak+1$, we first construct a surface of genus ak with k-fold rotational symmetry by the above construction. We can modify this surface to increase its genus by 1 while preserving its symmetry as follows. See the right of Fig. 2.2. The axis of a genus ak surface intersects the surface at two points—at the top and the bottom. Removing an invariant disk around each of these points creates two boundary components. Connecting the two boundary components with a cylinder yields an embedding of a surface of genus $ak+1$ with k-fold symmetry. □

By way of some elementary number theory, we show that all sufficiently large integers have either the form $ak + b(k-1)$ or the form $ak + 1$.

Lemma 2.2. *If $k \geq 5$ and $g \geq (k-1)(k-3)$, then g can either be written in the form $ak + b(k-1)$ with $a, b \in \mathbb{Z}_{\geq 0}$ or in the form $ak + 1$ with $a \in \mathbb{Z}_{>0}$.*

Proof. All integers at least $(k-1)(k-2)$ can be written in the form $ak + b(k-1)$ with $a, b \in \mathbb{Z}_{\geq 0}$ by the solution to the Frobenius coin problem. Further, every number from $(k-1)(k-3)$ to $k(k-3)$ can also be written as a sum of k's and $k-1$'s. Start with $k-3$ copies of $k-1$ and replace the $k-1$'s one at a time by k's. Finally, $k(k-3)+1 = (k-1)(k-2)-1$ is of the form $ak+1$. □

In addition to producing elements of order k in the stable range $g \geq (k-1)(k-3)$, we note that the construction given in Lemma 2.1 is also valid for approximately half of the values of g less than $(k-3)(k-1)$. Specifically, $(k^2-3k-4)/2$ of these k^2-4k+2 smaller values of g are either of the form $ak + b(k-1)$ or $ak+1$. This amount is simply $\sum_{i=3}^{k-2} i$, since $\{k-1, k, k+1\}$ is the first run of numbers of the given forms and $\{(k-4)(k-1), ..., (k-4)k+1)\}$ is the last run less than $(k-1)(k-3)$.

Also, note that Lemma 2.1 includes the cases where k is 2, 3, or 4. However, the construction we use to create the generating sets of Theorem 4.1 does not work for these small orders. However, these values of k are those already treated by Luo, Brendle and Farb, Kassabov, and Monden in their work on generating sets for $\mathrm{Mod}(S_g)$ consisting of elements of fixed finite order.

3. Building a Dehn twist

In this section, we show that a Dehn twist in $\mathrm{Mod}(S_g)$ about a nonseparating curve may be written as a product in four elements whenever these elements act on a small collection of curves in a specified way. In fact, even fewer than four elements will suffice as long as products in these elements act on the collection of curves as specified. In our proof, we follow the argument that Luo [15] gave for writing a Dehn twist as a product of involutions, as well as the pair swap argument made by Brendle and Farb [2].

We write T_c for the (left) Dehn twist about the curve c. Recall the *lantern relation* that holds among Dehn twists about seven curves arranged in a sphere with four boundary components, called a *lantern*. In the left of Fig. 3.1 we depict a lantern L that is a subsurface of S_g. Singling out this particular lantern is convenient for our proof of Theorem 4.1. Note that $S_g \setminus L$ is connected. Seven curves lie in L in a lantern arrangement, and several of these are Humphries curves. We will call these seven curves *lantern curves*. We have the following lantern relation:

$$T_{\alpha_1} T_{\alpha_2} T_{x_1} T_{\gamma_2} = T_{\gamma_1} T_{x_3} T_{x_2}.$$

Recall also that for a Dehn twist T_c and a mapping class f, we have $f T_c f^{-1} = T_{f(c)}$.

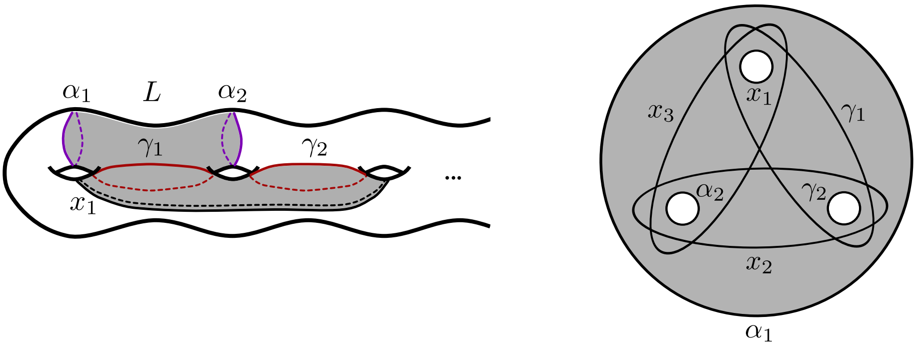

Fig. 3.1. On the left, the subsurface L and five of the lantern curves. Two lantern curves are omitted for clarity. On the right, the subsurface L and all seven lantern curves.

Lemma 3.1. *Suppose we are given the subsurface L in S_g and elements f, g, and h in* $\mathrm{Mod}(S_g)$ *such that*

$$f(\gamma_1) = \gamma_2$$
$$g(x_3, x_1) = (\gamma_1, \gamma_2)$$
$$h(x_2, \alpha_2) = (\gamma_1, \gamma_2).$$

Then the Dehn twist T_{α_1} may be written as a product in f, g, h, an element conjugate to f, and their inverses.

While this lemma is stated for a specific Dehn twist by way of a specific lantern, the result holds for other Dehn twists by the *change of coordinates principle*: if two collections of curves on a surface S_g are given by the same topological data, then there exists a homeomorphism of S_g to itself that maps the first collection of curves to the second. Details are given by Farb and Margalit [5].

Proof. Since Dehn twists about nonintersecting curves commute, one form of the lantern relation for L is

$$T_{\alpha_1} = (T_{\gamma_1} T_{\gamma_2}^{-1})(T_{x_3} T_{x_1}^{-1})(T_{x_2} T_{\alpha_2}^{-1}).$$

Applying the assumptions on the elements g and h yields

$$T_{\alpha_1} = (T_{\gamma_1} T_{\gamma_2}^{-1})(g^{-1}(T_{\gamma_1} T_{\gamma_2}^{-1})g)(h^{-1}(T_{\gamma_1} T_{\gamma_2}^{-1})h).$$

Applying the assumption on the element f and regrouping yields

$$\begin{aligned} T_{\alpha_1} &= ((f^{-1} T_{\gamma_2} f) T_{\gamma_2}^{-1})(g^{-1}(f^{-1} T_{\gamma_2} f) T_{\gamma_2}^{-1} g)(h^{-1}(f^{-1} T_{\gamma_2} f) T_{\gamma_2}^{-1} h) \\ &= (f^{-1}(T_{\gamma_2} f T_{\gamma_2}^{-1}))(g^{-1} f^{-1}(T_{\gamma_2} f T_{\gamma_2}^{-1})g)(h^{-1} f^{-1}(T_{\gamma_2} f T_{\gamma_2}^{-1})h). \end{aligned}$$

We have written T_{α_1} as a product in f, g, h, $T_{\gamma_2} f T_{\gamma_2}^{-1}$, and their inverses. □

Note that if f has order k, then so does $T_{\gamma_2} f T_{\gamma_2}^{-1}$ since it is a conjugate of f. Finally, notice that we required very little of f, g, and h in this argument—only that they map one specific curve or one specific pair of curves to another. We will take advantage of this flexibility in the proof of Theorem 4.1.

4. Generating $\mathrm{Mod}(S_g)$ with four elements of order k

In this section and in the following section we prove the two parts of the following theorem, which is our main technical result.

Theorem 4.1. *(1) Let $k \geq 5$ and let $g > 0$ be of the form $ak + b(k-1)$ with $a, b \in \mathbb{Z}_{\geq 0}$ or of the form $ak+1$ with $a \in \mathbb{Z}_{>0}$. Then $\mathrm{Mod}(S_g)$ is generated by four elements of order k.*

(2) Let $k \geq 8$ or $k = 6$ and let $g > 0$ be of the form $ak + b(k-1)$ with $a, b \in \mathbb{Z}_{\geq 0}$. Then $\mathrm{Mod}(S_g)$ is generated by three elements of order k. If instead $k = 7$ and g is of the form $7 + 7a + 6b$ with $a, b \in \mathbb{Z}_{\geq 0}$, then $\mathrm{Mod}(S_g)$ is generated by three elements of order 7.

Theorem 1.1 in the introduction follows directly from Theorem 4.1 along with Lemma 2.2 (for the case $g = 5$) and the observation that any $g \geq (k-1)^2 + 1$ may be written as a sum of k's and $(k-1)$'s with at least one summand equal to k.

In this section we prove the first part of Theorem 4.1 about generating with four elements of fixed finite order. In order to illustrate our construction, we depict the particular case $k = 5$ and $g = 18$ in Fig. 4.3. In what follows, a *chain* of curves on a surface is a sequence of curves $c_1, \ldots, c_t$ such that pairs of consecutive curves in the sequence intersect exactly once and each other pair of curves is disjoint.

Proof of Theorem 4.1, (1). We begin with the case where $g = ak + b(k-1)$ and treat the case where $g = ak + 1$ with a small modification at the end of the proof. Since $g = ak + b(k-1)$, we have a k-fold symmetric embedding of S_g in $\mathbb{R}^3$ as constructed in Lemma 2.1. Call this embedded surface Σ_g and let it be comprised of a surfaces of genus k followed by b surfaces of genus $k-1$. Let σ_1 through σ_{a+b} denote these k-symmetric subsurfaces of Σ_g. Let r be a rotation of Σ_g by $2\pi/k$ about its axis.

We will construct our desired elements by mapping S_g to Σ_g, performing a rotation r, and then mapping back to S_g. In doing so we will specify how individual curves map over and back again, and so control how curves are permuted among themselves. In order to construct these maps, it is convenient to label curves on S_g and Σ_g as follows. Take on the one hand the usual embedding of the Humphries curves in S_g as shown in Fig. 1.1 and the upper-left of Fig. 4.3. We will refer to the α_i, β_i and γ_i curves as *α curves*, *β curves*, and *γ curves*, respectively. The Humphries curves consist of a chain of $2g-1$ curves that alternate between β curves and γ curves as well as two additional α curves.

Similarly, take the k-fold symmetric embedded surface Σ_g and embed in each σ_i a chain of curves of length $2g_i - 1$, where g_i is the genus of σ_i. See Fig. 4.1 and the upper-left of Fig. 4.3. We label the curves in these chains also as β and γ curves and

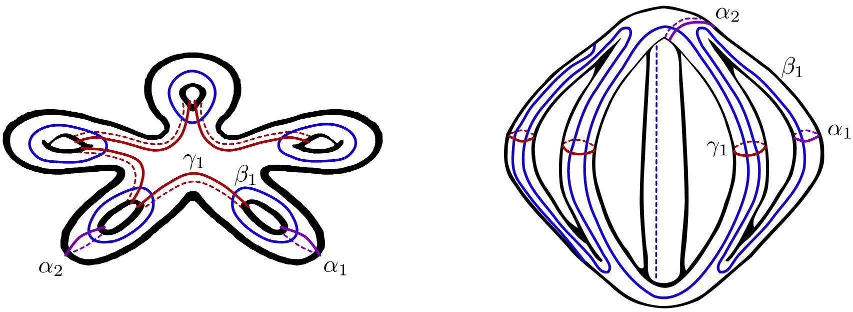

Fig. 4.1. The embeddings of chains of curves in the σ_i. The α curves are only included in the subsurface σ_1.

note that they are embedded so that $r(\beta_i) = \beta_{i+1}$, $1 \le i \le g_i - 1$, and $r(\gamma_i) = \gamma_{i+1}$, $1 \le i \le g_i - 2$. We will use these labels as "local coordinates"—saying, for instance, "the β_2 curve in σ_3." In σ_1 we additionally embed two α curves, α_1 and α_2, such that each respectively intersects β_1 and β_2 once, intersects no other curves, and $r(\alpha_1) = \alpha_2$.

We are now prepared to define three homeomorphisms $\hat{f}$, $\hat{g}$, and $\hat{h}$ from S_g to Σ_g. We will use these maps to define three homeomorphisms of the form $\hat{f}^{-1} r \hat{f}$ and will show that the corresponding mapping classes (1) have order k, (2) satisfy Lemma 3.1, and (3) generate a subgroup that puts the Humphries curves into the same orbit.

We first construct a homeomorphism $\hat{f}$. The β and γ Humphries curves in S_g form a chain of length $2g - 1$. By removing some of the γ curves from this chain, we form $a + b$ smaller chains. The first a chains will be $2k - 1$ curves long and the last b chains will be $2k - 3$ curves long. We accomplish this by removing every kth γ curve up to γ_{ak}, and then every $(k-1)$st γ curve thereafter. We call these the *excluded* γ curves. Call the resulting chains F_i, keeping their sequential order. We add to F_1 the curves α_1 and α_2.

Note that the curves in each F_i form a chain of simple closed curves in S_g and the union of the F_i is nonseparating. (Note that F_1 is not quite a chain because of the α_2 curve.) By the change of coordinates principle, there is a homeomorphism $\hat{f}$ that takes curves in the F_i to the curves in the chains in σ_i as specified above, as these chains of curves have the same length. (Recall that in σ_1 we have two additional curves that correspond to the α curves of S_g.) Let f be the mapping class of $\hat{f}^{-1} r \hat{f}$. Then f has order k and maps γ_1 to γ_2 as required by Lemma 3.1.

We now construct $\hat{g}$. We form triples of curves G_i, $2 \le i \le a + b$. To form each G_i, we take the second-to-last β curve in F_{i-1}, the excluded γ curve falling between F_{i-1} and F_i, and the second β curve in F_i.

Note that the curves in $\cup_i G_i$ are in the complement of L, that they are nonseparating simple closed curves, that they are disjoint, and that their union is nonseparating. By the change of coordinates principle, there is a homeomorphism $\hat{g}$ that maps the curves in L and the curves $\cup_i G_i$ to a collection of curves of the same topological type in Σ_g as follows:

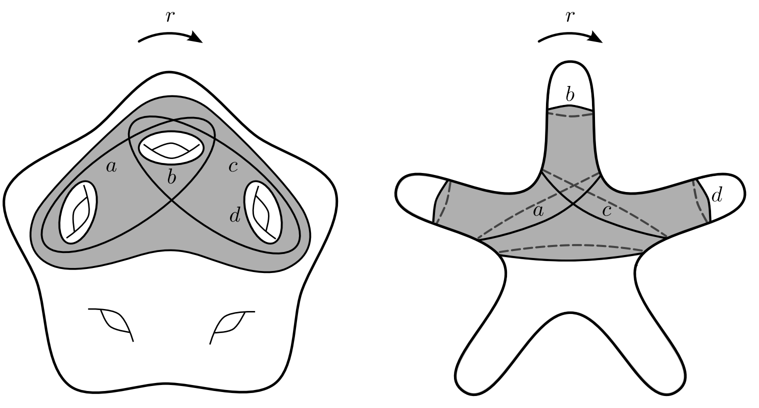

Fig. 4.2. The important curves of the subsurface L as embedded by $\hat{g}$ and $\hat{h}$ in σ_1 when the genus of σ_1 is k and when it is $k-1$. The latter is depicted as seen from above.

$$\begin{aligned} \hat{g}\colon S_g &\longrightarrow \Sigma_g \\ (x_3, x_1, \gamma_1, \gamma_2) &\longmapsto (a, b, c, d) \text{ in } \sigma_1 \text{ as in Fig. 4.2} \\ G_i &\longmapsto (\beta_1, \beta_2, \beta_3) \text{ in } \sigma_i,\ 2 \le i \le a+b \end{aligned}$$

Note that the specified image curves are of the same topological type as the four curves in L and the curves in $\cup_i G_i$. Note also that the embedding of the lantern curves depends on whether the genus of σ_1 is k or $k-1$; see Fig. 4.2. In both embeddings, the image of L is again nonseparating. Let g be the mapping class of $\hat{g}^{-1} r \hat{g}$. Then g has order k and maps the pair (x_3, x_1) to the pair (γ_1, γ_2) as required in Lemma 3.1.

Finally, we construct $\hat{h}$. We form pairs of curves H_i, $2 \le i \le a+b$. Each H_i consists of the first β curve and the second γ curve in F_i. The map $\hat{h}$ also specifies the mapping of the Humphries curve β_4. Let $\hat{h}$ be a homeomorphism that maps curves as follows:

$$\begin{aligned} \hat{h}\colon S_g &\longrightarrow \Sigma_g \\ (\gamma_1, \gamma_2, x_2, \alpha_2) &\longmapsto (a, b, c, d) \text{ in } \sigma_1 \text{ as in Fig. 4.2} \\ \beta_4 &\longmapsto r(d) \text{ in } \sigma_1 \\ H_i &\longmapsto (\beta_1, \beta_2) \text{ in } \sigma_i,\ 2 \le i \le a+b \end{aligned}$$

Let h be the mapping class of $\hat{h}^{-1} r \hat{h}$. Then h has order k and h^{-1} maps the pair (x_2, α_2) to the pair (γ_1, γ_2) as required by Lemma 3.1. (We use h^{-1} here because we want all of our generators to be conjugate and because the lantern curves are in a fixed cyclic order.)

We now show that the Humphries curves are in the same orbit under $\langle f, g, h \rangle$. Refer to Fig. 4.4. First, note that every β and γ Humphries curve in S_g is in some F_i or G_i. Additionally, powers of f map any β curve in F_i to any other β curve in the same F_i, and likewise for γ curves. Call these orbits of curves $F_i\beta$ and $F_i\gamma$. In the same way,

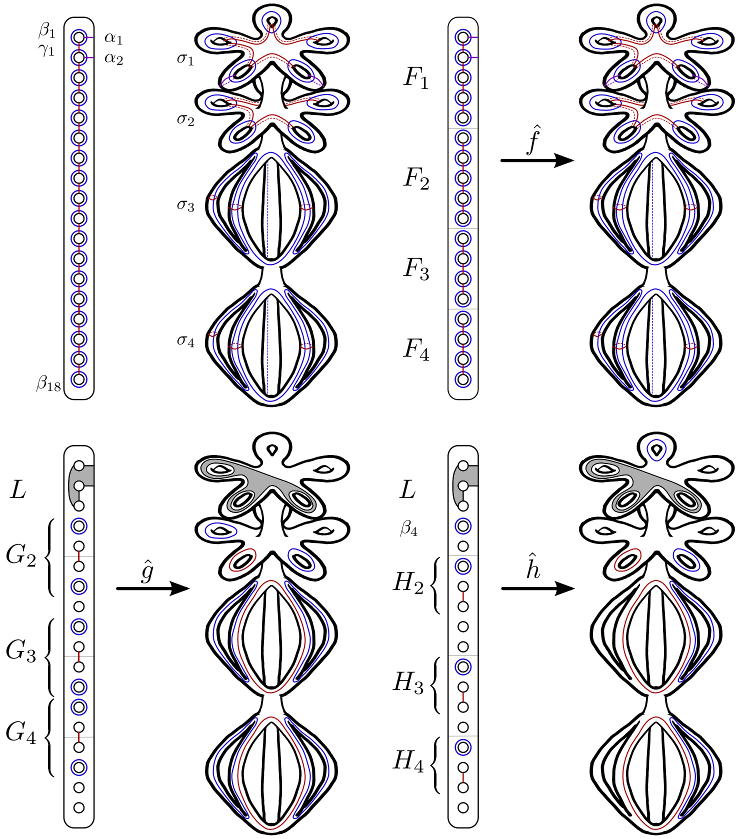

Fig. 4.3. The Humphries curves in S_{18}. Σ_{18} with "local coordinate" curves in each σ_i. The curves in the F_i, G_i, H_i, and the subsurface L, along with their images under $\hat{f}$, $\hat{g}$, and $\hat{h}$.

a power of g maps any curve in G_i to any other curve in the same G_i. Thus at most we have the following orbits of the Humphries curves under $\langle f, g, h\rangle$: the $F_i\beta$, the $F_i\gamma$, the G_i, α_1, and α_2. We will show that these are all in fact a single orbit under $\langle f, g, h\rangle$.

The element f maps α_1 to α_2 when σ_1 has genus k and maps α_1 to γ_1 when σ_1 has genus $k-1$. The element g maps the lantern curve γ_2 to the lantern curve α_2. Thus each of α_1 and α_2 is in the same orbit as some γ curve.

A power of g takes a curve in $F_i\beta$ to a curve in $F_{i-1}\beta$ as well as to a curve in G_i, $2 \leq i \leq a+b$. Additionally, a power of h takes a curve in $F_i\beta$ to a curve in $F_i\gamma$, $1 \leq i \leq a+b$. (Note that in the case of F_1, we have $h^2(\gamma_2) = \beta_4$.) Thus all Humphries curves are in a single orbit under $\langle f, g, h\rangle$. By Lemma 3.1, the Dehn twist about α_1 may be written as a product in f, g, h, and $T_{\gamma_2} f T_{\gamma_2}^{-1}$. Thus all Dehn twists about the Humphries curves may be written as products in our four elements of order k, and so they generate $\mathrm{Mod}(S_g)$.

In the case where $g = ak+1$, we may modify the construction to show that $\mathrm{Mod}(S_g)$ is again generated by four elements of order k. Take a connect sum of a

$$\begin{tikzcd}
 & G_2 \arrow[d, leftrightarrow, "g"] & G_3 \arrow[d, leftrightarrow, "g"] & \\
F_1\beta \arrow[r, leftrightarrow, "g"] \arrow[d, leftrightarrow, "h"] & F_2\beta \arrow[r, leftrightarrow, "g"] \arrow[d, leftrightarrow, "h"] & F_3\beta \arrow[r, leftrightarrow, "g"] \arrow[d, leftrightarrow, "h"] & \dots \\
F_1\gamma \arrow[d, leftrightarrow, "f\ (a{=}0)"'] \arrow[dr, leftrightarrow, "g"] & F_2\gamma & F_3\gamma & \\
\alpha_1 \arrow[r, leftrightarrow, "f\ (a{>}0)"'] & \alpha_2 & &
\end{tikzcd}$$

Fig. 4.4. Each node is a collection of curves that are in the same orbit under the subgroup generated by a single element. Each arrow indicates when a power of an element maps a curve in one collection to a curve in another. Since every Humphries curve is in at least one of the collections, all Humphries curves are in the same orbit under the subgroup $\langle f, g, h\rangle$.

surfaces of genus k and insert one further handle along the axis of rotation, as in Lemma 2.1. The element r is a rotation of this embedded surface by $2\pi/k$ and f is defined as above by ignoring the final two Humphries curves β_g and γ_{g-1}. We must modify our other elements of order k so that they place these two additional Humphries curves into the same orbit as all of the other Humphries curves under the subgroup $\langle f, g, h\rangle$. Modify $\hat{g}$ so that it additionally maps β_g to $r(d)$ in σ_1 and modify $\hat{h}$ so that it additionally maps γ_{g-1} to $r^2(d)$ in σ_1. These modifications preserve the fact that the curves involved are disjoint and that their union is nonseparating. The elements g and h now put the curves β_g and γ_{g-1} into the same orbit as the other Humphries curves. Hence $\mathrm{Mod}(S_g)$ is also generated by four elements of order k when $g = ak + 1$. $\square$

5. Sharpening to three elements

In this section we prove the second part of Theorem 4.1.

Proof of Theorem 4.1, (2). We first provide the construction for the cases $k \geq 8$ and then afterwards give the constructions for $k = 7$ and $k = 6$. Let $k \geq 8$. By assumption we may write g in the form $ak+b(k-1)$. We construct the homeomorphism $\hat{f} : S_g \to \Sigma_g$ as in the proof of the first part of theorem, except with the modification that it additionally maps the α curve that intersects the final β curve in F_1 (called α_ℓ) to the curve $r^{-1}\hat{f}(\alpha_1)$. See Fig. 5.1. We again let f be the mapping class of $\hat{f}^{-1}r\hat{f}$, and f has order k.

We now construct $\hat{g}$. Let G_2 consist of α_ℓ, the excluded γ curve falling between F_1 and F_2, the first γ curve in F_2, and the third β curve in F_2. For $2 < i \leq a + b$, let G_i be the last γ curve in F_{i-1}, the excluded γ curve between F_{i-1} and F_i, the first γ curve in F_i, and the third β curve in F_i. See Fig. 5.2. Let $\hat{g}$ be a homeomorphism that maps the specified curves as follows:

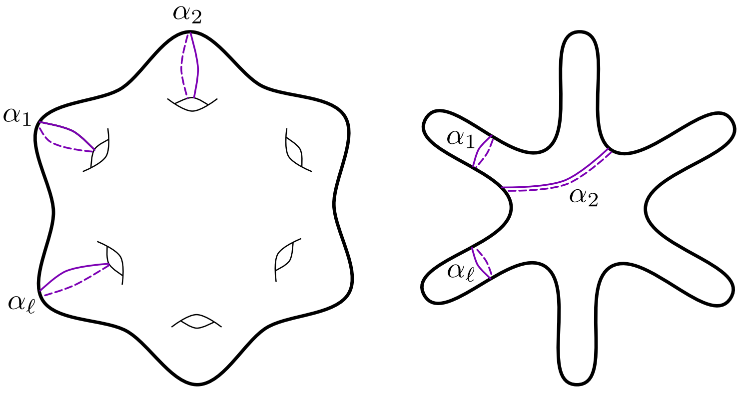

Fig. 5.1. The images of α curves embedded in σ_1 by $\hat{f}$. With this embedding, the rotation r maps α_ℓ to α_1.

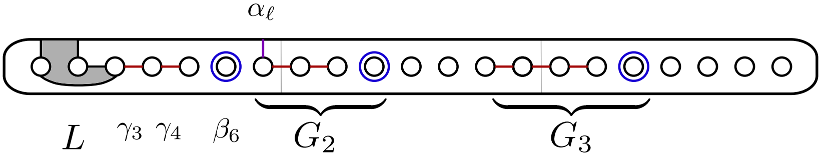

Fig. 5.2. The curves mapped by $\hat{g}$ in the case $k = 8$, $g = 21$. This is a worst case example where k has the smallest possible value and all of the σ_i have genus $k - 1$.

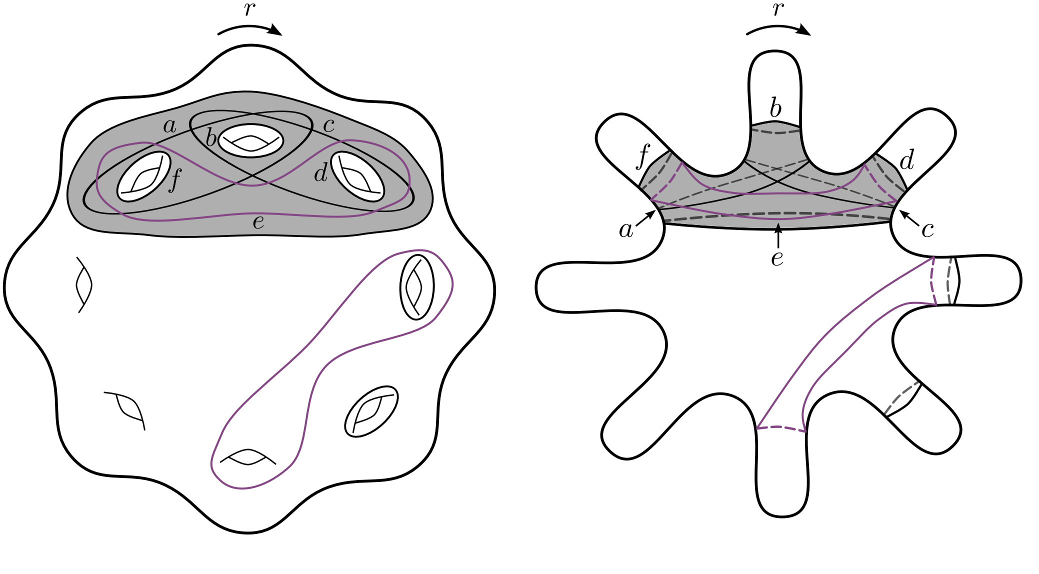

Fig. 5.3. The embedding of the subsurface L in σ_1 when the genus of σ_1 is k and when it is $k - 1$. Also, the embedding of γ_3, γ_4, and β_6. These diagrams depict the case $k = 8$.

$$\begin{tikzcd}
 & & G_2 \arrow[d, leftrightarrow, "g"] & G_3 \arrow[d, leftrightarrow, "g"] & \\
 & F_1\beta \arrow[d, leftrightarrow, "g"] & F_2\beta \arrow[d, leftrightarrow, "g"] & F_3\beta \arrow[d, leftrightarrow, "g"] & \\
 & F_1\gamma \arrow[d, leftrightarrow, "f\ (a=0)"] & F_2\gamma \arrow[r, leftrightarrow, "g"] \arrow[d, leftrightarrow, "g"] & F_3\gamma \arrow[r, leftrightarrow, "g"] & \dots \\
\alpha_2 \arrow[ur, leftrightarrow, "h=f^{-2}g^3"] \arrow[r, leftrightarrow, "f\ (a>0)"'] & \alpha_1 \arrow[r, leftrightarrow, "f"] & \alpha_l & &
\end{tikzcd}$$

Fig. 5.4. Again, each node is a collection of curves that are in the same orbit under the subgroup generated by a single element. Each arrow indicates when a power of an element maps a curve in one collection to a curve in another. Since every Humphries curve is in at least one of the collections, all Humphries curves are in the same orbit under $\langle f, g\rangle$.

$$\begin{aligned}
\hat{g}\colon S_g &\longrightarrow \Sigma_g \\
(x_3, x_1, \gamma_1, \gamma_2, x_2, \alpha_2) &\longmapsto (a, b, c, d, e, f) \text{ as in Fig. 5.3} \\
(\gamma_3, \gamma_4) &\longmapsto (r^3(e), r^3(f)) \\
\beta_6 &\longmapsto r^4(f) \text{ in } \sigma_1 \\
G_i &\longmapsto (\beta_1, \beta_2, \beta_3, \beta_4) \text{ in } \sigma_i,\ 2 \le i \le a+b
\end{aligned}$$

Let g be the mapping class of $\hat{g}^{-1}r\hat{g}$. Then g has order k and maps the pair (x_3, x_1) to the pair (γ_1, γ_2) as required by Lemma 3.1. Additionally, $g^3(x_2, \alpha_2) = (\gamma_3, \gamma_4)$ and $f^{-2}(\gamma_3, \gamma_4) = (\gamma_1, \gamma_2)$. We may therefore define $h = f^{-2}g^3$ so that h satisfies the hypothesis of Lemma 3.1, as $h(x_2, \alpha_2) = (\gamma_1, \gamma_2)$. Thus the Dehn twist about α_1 may be written as a product in f, g, and $T_{\gamma_2} f T_{\gamma_2}^{-1}$.

Finally, we show that all of the Humphries curves are in the same orbit under $\langle f, g\rangle$, as can be seen in Fig. 5.4. Again, every β and γ Humphries curve in S_g is in some $F_i\beta$, $F_i\gamma$, or G_i. Therefore we have at most the following orbits of the Humphries curves under $\langle f, g\rangle$: the $F_i\beta$, the $F_i\gamma$, the G_i, α_1, and α_2. We will show that these are all in fact a single orbit under $\langle f, g\rangle$. For $i > 2$, powers of g put the curves in G_i, $F_i\beta$, $F_i\gamma$, and $F_{i-1}\gamma$ in the same orbit. Note that powers of g map β_6 to γ_2 and a γ curve in F_2 to α_ℓ, while the product h carries α_2 to γ_2. Also, f maps α_ℓ to α_1 and maps α_1 either to α_2 or γ_1, depending on the genus of σ_1. Considering this, all of the curves are in the same orbit under the subgroup $\langle f, g\rangle$. Therefore the Dehn twist about each of the Humphries curves may be written as a product in the three elements f, g, and $T_{\gamma_2} f T_{\gamma_2}^{-1}$, and so they generate $\mathrm{Mod}(S_g)$.

In the case where $k = 7$, the same construction as above goes through as long as the genus of σ_1 is 7. As illustrated in Fig. 5.2, under this assumption there is enough room to configure all of the required curves in the construction of $\hat{g}$. The hypotheses of the theorem in this case exactly demand that the genus of σ_1 be 7.

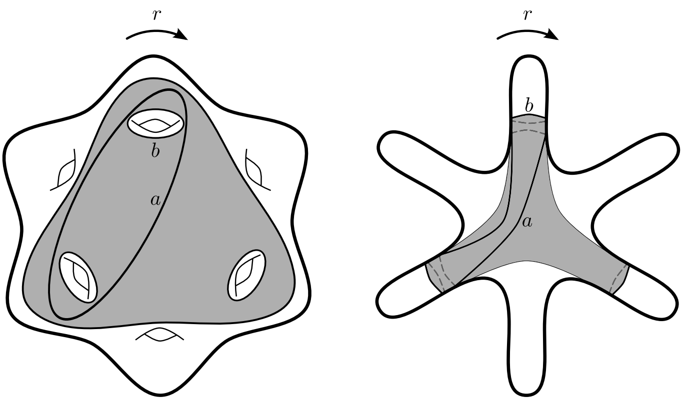

Fig. 5.5. The embeddings of the subsurface L in σ_1 when $k = 6$.

In the case where $k = 6$, we use the same construction as above for f and construct the element g as follows, exploiting the three-fold symmetry of a lantern. See Fig. 5.5. Let $\hat{g}$ be a homeomorphism that maps the specified curves as follows:

$$\begin{aligned} \hat{g}\colon S_g &\longrightarrow \Sigma_g \\ (x_3, x_1, \gamma_1, \gamma_2, x_2, \alpha_2) &\longmapsto (a, b, r^2(a), r^2(b), r^4(a), r^4(b)) \text{ as in Fig. 5.5} \\ \beta_4 &\longmapsto r(b) \text{ in } \sigma_1 \\ G_i &\longmapsto (\beta_1, \beta_2, \beta_3, \beta_4) \text{ in } \sigma_i,\ 2 \le i \le a+b \end{aligned}$$

In this case, g^2 and g^4 play the roles of g and h in Lemma 3.1, and all Humphries curves are again in the same orbit. □

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