UNIVERSAL BOUNDS FOR TORSION GENERATING SETS OF MAPPING CLASS GROUPS

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Abstract. We show that there exists a universal upper bound on the number of conjugates of a periodic mapping class of order at least 3 required to generate $\mathrm{Mod}(S_g)$ for all $g \geq 3$. This upper bound is independent of both g and the element chosen, and our proof shows that this upper bound may be taken to be 60. We also give upper and lower bounds on the number of conjugates of a given involution required to generate $\mathrm{Mod}(S_g)$ for $g \geq 3$. For involution normal generators, the supremum of these lower bounds is infinite. These results answer Problem 4.3 in Margalit's problem list.

1. Introduction

Let S_g denote the connected, closed, orientable surface of genus g. The mapping class group $\mathrm{Mod}(S_g)$ is the group of homotopy classes of orientation-preserving homeomorphisms of S_g.

The author and Margalit proved the following theorem.

Theorem 1.1. [12, Theorem 1.1] *For every $g \geq 3$, every nontrivial periodic mapping class that is not a hyperelliptic involution normally generates $\mathrm{Mod}(S_g)$.*

At the time this theorem was proven, several examples of small all-conjugate torsion generating sets for $\mathrm{Mod}(S_g)$ were known; see below for a discussion. Based on Theorem 1.1 and this corroborating evidence, the author and Margalit raised the following question. It was also recorded as Problem 4.3 by Margalit in his problems paper [13].

Question 1.2. [12, Question 3.4] *Is there a number N, independent of g, so that if f is a periodic normal generator of $\mathrm{Mod}(S_g)$ then $\mathrm{Mod}(S_g)$ is generated by N conjugates of f?*

With Theorem 1.1 in hand, it is not difficult to give an upper bound on the number of conjugates required that is linear in g. For instance, $24g + 12$ conjugates suffice, which follows from our Lemma 3.1 and the result that $\mathrm{Mod}(S_g)$ is generated by $2g + 1$ Dehn twists about non-separating curves. Note that this same result of Humphries also shows that there is no universal upper bound on the size of Dehn twist generating sets [6].

The results in this paper resolve Question 1.2.

For the class of involutions—and therefore for the case of general periodic normal generators—we show that there is no universal bound on the number of conjugates of f required to generate $\mathrm{Mod}(S_g)$.

Theorem 1.3. *There does not exist a number N, independent of g, so that if f is a periodic normal generator of $\mathrm{Mod}(S_g)$ with $|f| = 2$ then $\mathrm{Mod}(S_g)$ is generated by N conjugates of f.*

This result should be viewed in contrast with the results by Brendle–Farb, Kassabov, Korkmaz, and Yildiz giving involution generating sets for $\mathrm{Mod}(S_g)$ of universally bounded size [1, 7, 8, 15]. For these results, the involutions in a given generating set need not all be conjugate, and the conjugacy classes used are hand-picked, rather than arbitrary.

On the other hand, we show that involutions are the exceptional case: there does exist a universal upper bound under the assumption that $|f| \geq 3$.

Theorem 1.4. *There a number N, independent of g, so that if f is a periodic normal generator of* $\mathrm{Mod}(S_g)$ *with* $|f| \geq 3$ *then* $\mathrm{Mod}(S_g)$ *is generated by N conjugates of f.*

Our proof of this theorem shows that N may be taken to be 60. In order to treat cases uniformly and to simplify arguments, we have not taken pains to optimize this bound. We can of course ask what the sharp universal upper bound may be under the hypothesis $|f| \geq 3$; we know of no obstacle to it being $N = 2$.

Prior results. There have been many results about generating mapping class groups with torsion; an overview is given in an earlier paper of the author [11]. In terms of giving upper bounds on the sizes of torsion generating sets for $\mathrm{Mod}(S_g)$ consisting of conjugate elements, the following results were previously known. Korkmaz showed that two conjugate elements of order $4g+2$ generate $\mathrm{Mod}(S_g)$ for $g \geq 3$ [9]. The author showed that three conjugate elements of order $k \geq 6$ generate $\mathrm{Mod}(S_g)$ for $g \geq (k-1)^2 + 1$ [11]; all of the elements used in the constructions of that paper can be realized as rotations of an embedding of S_g in $\mathbb{R}^3$. Finally, Yoshihara proved that three conjugate elements of order 6 generate $\mathrm{Mod}(S_g)$ for $g \geq 10$ [16].

Outline. In Section 2 we prove our results about generating sets for $\mathrm{Mod}(S_g)$ comprised of conjugates of an involution. In Section 3 we describe our proof strategy for Theorem 1.4, prove several preliminary technical lemmas, and then apply these to prove Theorem 1.4.

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2. Involution generating sets

Every pair (r, s) of non-negative integers determines a conjugacy class of involution homeomorphisms on S_g where $g = 2r + s$; we denote a representative of this class by $i_{r,s}$. The involution $i_{r,s}$ rotates r pairs of handles to swap them and "skewers" s handles, as illustrated in Figure 2.1. Pairs (r, s) with $g = 2r + s$ in fact parametrize conjugacy classes of involutions in $\mathrm{Mod}(S_g)$ for $g \geq 2$, a classification that goes back to the work of Klein; see, for instance, the survey by Dugger [3]. An involution $i_{r,s}$ induces an action on $H_1(S_g; \mathbb{Q})$ that preserves $2s$ subspaces of dimension 1 and induces involutions on r pairs of subspaces of dimension 2.

The intersection of the preserved subspaces of the generators of a group action is also preserved by the group. Consequently, any generating set for $\mathrm{Mod}(S_g)$ comprised of conjugates of $i_{r,s}$ must contain least $(2r+s)/2r$ generators, since $\mathrm{Mod}(S_g)$ acts transitively on 1-dimensional subspaces of $H_1(S_g; \mathbb{Q})$. The quantity $(2r + s)/2r$ is arbitrarily large whenever we let s be sufficiently large compared to r.

Note that $i_{0,g}$ is a hyperelliptic involution in $\mathrm{Mod}(S_g)$ and it is not a normal generator of $\mathrm{Mod}(S_g)$; for $i_{0,g}$, the quantity $(2r + s)/2r$ is undefined.

While $(2r + s)/2r$ is a lower bound on the number of conjugates of $i_{r,s}$ required to generate $\mathrm{Mod}(S_g)$, we can also give an easy upper bound. This will also show that with the constraint $(2r + s)/2r < n$, there exists a universal bound $N(n)$ on the number of conjugates of $i_{r,s}$ required to generate $\mathrm{Mod}(S_g)$. Our argument additionally serves as a warm-up for our proof of Theorem 1.4.

Let H_g be the set of curves $\{a_1, \ldots, a_{2g}, b\}$ corresponding to the Humphries generating set, where the curves a_i form a chain of length $2g$; the curves H_g are depicted in Figure 2.2. Each generator that is a conjugate of $i_{r,s}$ puts $2r$ disjoint non-separating curves into r 2-cycles, and these $2r$ curves are collectively non-separating. Therefore in a subgroup

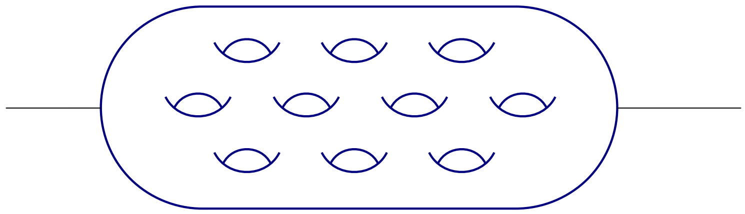

FIGURE 2.1. A schematic for the involution $i_{r,s}$, where an involution $i_{3,4}$ on S_{10} is illustrated.

generated by $k = \lceil g/2r \rceil = \lceil (2r+s)/2r \rceil$ generators, we can ensure that each of g disjoint curves is in a 2-cycle, and that these are collectively non-separating. In the subgroup generated by k additional generators, the same holds true for all $2g$ curves in $H_g - \{b\}$. We may add a further $2k-1$ generators so that the curves in $H_g - \{b\}$ are all in the same orbit under the generated subgroup, and adding one further generator ensures that b is also in the same orbit. So far this is a total of $4k$ generators. Since T_b can be written as a product in at most an additional 6 generators by the lantern relation trick (see Lemma 3.1), we have that all Dehn twists about the curves in H_g are in the generated subgroup, which therefore equals $\mathrm{Mod}(S_g)$. Summing up, we have that $4k + 6 = 4 \cdot \lceil (2r+s)/2r \rceil + 6$ generators suffice.

We record the results of this section in the following theorem.

Theorem 2.1. *Let $g \geq 3$ and let $i_{r.s}$ be an involution in $\mathrm{Mod}(S_g)$ that is not a hyperelliptic involution. Any generating set for $\mathrm{Mod}(S_g)$ consisting of conjugates of the involution $i_{r,s}$ contains at least $(2r+s)/2r$ generators. Further, $4 \cdot \lceil (2r+s)/2r \rceil + 6$ conjugates of $i_{r,s}$ suffice to generate $\mathrm{Mod}(S_g)$.*

This result immediately implies Theorem 1.3, stated in the introduction.

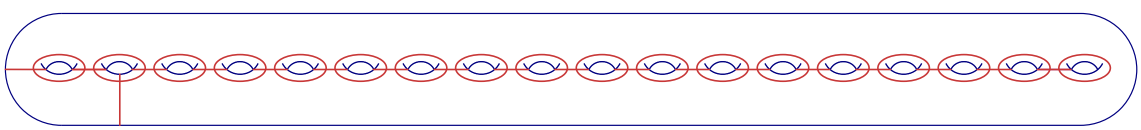

FIGURE 2.2. The set of curves $H_g = \{a_1, \ldots, a_{2g}, b\}$ corresponding to the Humphries generating set.

3. A UNIVERSAL BOUND FOR NON-INVOLUTION TORSION GENERATING SETS

In this section we prove our main result, Theorem 1.4. We begin by outlining our proof strategy, which breaks up into three steps. We then prove the lemmas that carry out the first two steps and then conclude with our proof of Theorem 1.4, which carries out the final step.

Proof strategy. The proof of Theorem 1.4 follows the same basic strategy taken by the author in his article [11], which gives a sharper version of Theorem 1.4 for a certain class of periodic elements (ones that can be represented by rotations of S_g embedded in $\mathbb{R}^3$) under the further assumption that $|f| \geq 5$. It is also the strategy used by other authors in proving results concerning small torsion generating sets for $\mathrm{Mod}(S_g)$, as well as in the proof of Theorem 2.1 above. The challenge is in carrying out the strategy for arbitrary periodic elements.

To prove Theorem 1.4, it suffices to show that each of the generators in the Humphries generating set of $2g+1$ Dehn twists about nonseparating curves lies in a subgroup generated by 60 conjugates of an arbitrary periodic element f with $|f| \geq 3$. Again, call the set of Humphries curves H_g; it consists of a chain of curves $a_1, \ldots, a_{2g}$ and an additional curve b. See Figure 2.2. By passing to powers, it suffices to consider f where $|f|$ is either 4 or an odd prime.

The proof then consists of three steps. First, using the lantern relation trick, we show that a single Dehn twist about a non-separating curve can be written as a product in at most 12 distinct conjugates of f. We show this as Lemma 3.1.

It remains to show that the curves in H_g lie in a single orbit under the action of a subgroup generated by at most 48 conjugates of f. As the second step, we show that there exists a maximal non-separating chain C of $2g$ curves in S_g such that f acts on a subset C^* of C of size at least $g/4$ where every orbit under $\langle f \rangle$ that contains an element of C^* in fact contains at least two elements of C^*. That is, we show that f shuffles a nice collection of curves, of size a definite fraction of g, in a nice way. This is shown for irreducible elements in Lemma 3.3 and for reducible elements in Lemma 3.4. We conclude by proving the main result, which involves showing that at most 48 conjugates of f generate a subgroup of $\mathrm{Mod}(S_g)$ that acts transitively on H_g; along with the 12 conjugates of f that generate a subgroup that contains T_b, this yields a generating set for $\mathrm{Mod}(S_g)$ consisting of at most 60 conjugates of f.

Step 1. Our first lemma is a straightforward consequence of the work in Sections 2 and 3 of the author's paper with Margalit [12].

Lemma 3.1. *Let $g \geq 3$ and let $f \in \mathrm{Mod}(S_g)$ be a periodic mapping class with $|f| \geq 3$. Then the Dehn twist about any fixed non-separating curve in S_g can be written as a product in at most 12 distinct conjugates of f.*

Proof. By the proof of Theorem 1.1 of [12], there exists a non-trivial power f^k of f and a curve c in S_g such that

(1) c is non-separating and c and $f^k(c)$ are disjoint and non-homologous,
(2) c is non-separating and c and $f^k(c)$ intersect exactly once, or
(3) c is separating and c and $f^k(c)$ are disjoint.

The third case implies the existence of a non-separating curve d such that d and $f^k(d)$ are disjoint and non-homologous, and so reduces to the first case. The second case implies that there is a conjugate h of f such that c and $h^k f^k(c)$ are disjoint and non-homologous, and so reduces to the first case at the cost of doubling the number of conjugates of f required. Finally, using the lantern relation trick, the first case shows that a single Dehn twist about a non-separating curve can be written as a product of 6 conjugates of f. (See for instance [4, Theorem 7.16].) So considering all cases, a single Dehn twist about a given non-separating curve can be written as a product in 12 conjugates of f. □

Step 2. We now proceed to the main technical work of the paper, where we guarantee that a definite fraction, independent of g, of the $2g+1$ curves in H_g can be shuffled around by a periodic element f with $|f| \geq 3$. We begin by classifying the irreducible mapping classes of odd prime order and of order 4.

Lemma 3.2. *Let p be either an odd prime or 4. Then there exists a unique $g(p) \geq 1$ such that $\mathrm{Mod}(S_{g(p)})$ contains an irreducible element of order p: $g(4) = 1$ and $g(p) = \frac{p-1}{2}$ when*

p is an odd prime. Up to conjugacy and powers, $\mathrm{Mod}(S_{g(p)})$ *contains exactly 1 such element when $p = 4$, which has signature $(4, 0; (1, 2), (1, 4), (1, 4))$; and finitely many such elements when p is an odd prime, each with signature of the form $(p, 0; (c_1, p), (c_2, p), (c_3, p))$, with $0 < c_i < p$.*

Proof. Let f be an irreducible periodic mapping class of order p, with p equaling either an odd prime or 4. By a result of Gilman [5], any irreducible periodic mapping class has as its quotient orbifold a sphere with three marked points. Since the index of each marked point is greater than 1 and must divide p, each index must equal p when p is prime and must equal 2 or 4 when p is 4. By the Riemann–Hurwitz formula, we have when p is an odd prime

$$2g - 2 = p(0 - 2) + 3(p - 1)$$

and so $g(p) = \frac{p-1}{2}$.

Similarly, when $p = 4$, the possible triples of indices are $\{2, 2, 4\}$, $\{2, 4, 4\}$, and $\{4, 4, 4\}$, since the LCM of the indices must be 4. We have, respectively,

$$2g - 2 = 4(0 - 2) + 7$$

$$2g - 2 = 4(0 - 2) + 8$$

$$2g - 2 = 4(0 - 2) + 9$$

Only the second equation, corresponding to $\{2, 4, 4\}$, yields a natural number for g, and so we have $g(4) = 1$.

The conjugacy class of a periodic element (that is not a free action) is determined by its signature $(n, g_0; (c_1, n_1), \ldots, (c_\ell, n_\ell))$. For our irreducible element of order p, we have either

$$(p, 0; (c_1, p), (c_2, p), (c_3, p))$$

when p is an odd prime and

$$(4, 0; (c_1, 2), (c_2, 2), (c_3, 4))$$

when $p = 4$, with $0 < c_i < p$. In each case, there is a requirement that $\sum_i c_i = 0 \pmod{p}$. We see that there are finitely many combinations. For $p = 4$, we have that there are two conjugacy classes: $(4, 0; (1, 2), (1, 4), (1, 4))$ and $(4, 0; (1, 2), (1, 4), (1, 4))$. These are powers of each other. □

The next two lemmas form the main technical result of this paper. Under the hypothesis that $|f|$ is an odd prime or 4, we give a topological descriptions of f and use this to show that f shuffles a definite fraction of the curves in a maximal non-separating chain in S_g. The lemmas treat the irreducible and reducible cases in turn. In the reducible case, our decomposition applies work of Gilman [5] and follows the outline of a geometric decomposition for general periodic elements given by Parsad–Rajeevsarathy–Sanki [14]; see especially their Theorem 2.27. Our decomposition also resembles the construction of periodic elements of prime order in the thesis of Chrisman, although Chrisman's goal is to produce periodic homeomorphisms that realize any given number of branch points [2, Chapter 3]. (That is, Chrisman gives a construction realizing each possible number of branch points for prime order elements, but he does not show that every prime order conjugacy class can be built in his manner.)

Lemma 3.3. *Let p be either an odd prime or 4. Let $g = g(p)$ and let $f \in \mathrm{Mod}(S_g)$ be an irreducible element of order p. Then there exists a set of curves C^* in S_g such that*

- *C^* is a subset of a maximal non-separating chain of curves in S_g,*
- *$|C^*| \geq 2g/5$, (alternatively, $|C^*| \geq p/4 - 1$ for $p \geq 13$),*
- *every orbit of curves under the action of $\langle f \rangle$ that contains a curve in C^* contains at least two curves in C^*, and*
- *each subset of curves in C^* that lies in the same orbit under $\langle f \rangle$ forms either a chain of length at least 2 or else consists of disjoint curves.*

In particular, when p is 3, 4, 5, 7, or 11, we may take C^* so that $|C^*|$ is 2, 2, 2, 3, or 3, respectively.

Proof. We first treat the case $p = 4$. Then S_g is the torus T and f can be realized by a rotation of a square torus by $\pi/2$ or $3\pi/2$. In either case, f exchanges the meridian and longitude of T, which form a chain of length 2. We take this chain to be C^*. We have that $|C^*| = 2 > 1 = g$.

Next we let $p \geq 13$ be an odd prime; we treat the cases $p \in \{3, 5, 7, 11\}$ afterwards. Since f has a fixed point, by a result of Kulkarni, f can be represented as a rotation of a polygon with an appropriate side pairing [10, Theorem 2]. Parsad–Rajeevsarathy–Sanki give an explicit construction for Kulkarni's existence result [14, Theorem 2.10]. In particular, they show that f can be realized by a rotation of a $2p$-gon D by $2\pi m/p$, where $(m, p) = 1$. The sides d_i of D are identified in pairs (in an orientation-preserving way) according to the formula $d_i \sim d_{i+k}$, where indices are taken mod $2p$ and where k is a constant that depends on the signature for f. In other words, each pair of sides that is identified is at a fixed distance k apart. From the formulas in the theorem of Parsad–Rajeevsarathy–Sanki, we see that k is necessarily odd. Further, we have that $k > 1$, since otherwise the resulting surface is a sphere.

Consider three cases, depending on whether k is 0, 1, or 2 (mod 3). Whenever $k \geq 6$, the curves may be chosen so that they are "interleaved"; thus the density of the packing depends only on the residue class (mod 3). In each case we will construct a set of curves C^* so that $|C^*| \geq 2g/5$. In each case, we will select curves from an orbit of curves c_i under $\langle f \rangle$, $1 \leq i \leq p$, where c_1 is formed by taking the union of two segments, each connecting midpoints of a pair of edges: d_1 and d_3, and d_{1+k} and d_{3+k}. The fact that $k \geq 3$ ensures that this union of segments is a simple closed curve. Each of the curves c_i is non-separating, and we will take a collection that forms a subset of a maximal non-separating chain; in fact, all of the curves in the collection be disjoint. Further, these curves are all in the same orbit, and so satisfy the third and fourth properties of C^* in the statement.

When $k = 0$ (mod 3), we may take $\lfloor 2p/6 \rfloor$ of the c_i so that they are disjoint and together do not separate. This case is illustrated in Figure 3.1 in a case where $p = 13$ and $\lfloor 2p/6 \rfloor = 4$.

When $k = 1$ (mod 3), we may take $\lfloor 2p/7 \rfloor$ of the c_i so that they are disjoint and together do not separate.

When $k = 2$ (mod 3), we may take $\lfloor 2p/8 \rfloor$ of the c_i so that they are disjoint and together do not separate.

In each case, we have that $|C^*| = \lfloor 2p/8 \rfloor = \lfloor (2g+1)/4 \rfloor \geq 2g/5$ and also $|C^*| \geq p/4 - 1$. For all odd primes $p \geq 13$ we have that $|C^*| \geq 2$ and that all of the curves in C^* belong to the same orbit, as desired.

For $p \in \{3, 5, 7, 11\}$, a more careful analysis is required. In these cases the bound $|C^*| \geq p/4 - 1$ is insufficient, since we require $|C^*| \geq 2$. We give a modified construction for C^* in each case.

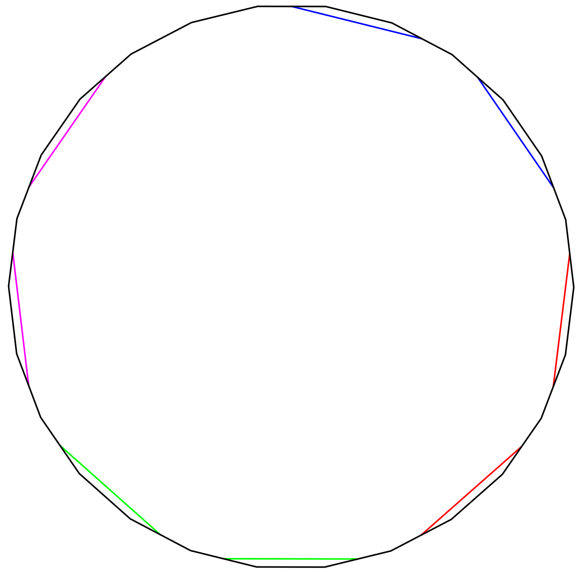

FIGURE 3.1. The collections of curves C^* for an irreducible periodic mapping class of order 13 for which $k = 3$.

When $p = 3$, $g(p) = 1$. Up to conjugacy and powers there is a single element to consider, corresponding to the signature $(3, 0; (1, 3), (1, 3), (1, 3))$. We have that D is a hexagon with opposite sides identified, and f is a rotation by $2\pi/3$. Then we may take for C^* a chain of 2 curves that are in the same orbit under f, namely, 2 curves corresponding to segments connecting midpoints of opposite sides of D.

When $p = 5$, $g(p) = 2$. Up to conjugacy and powers there is a single element to consider, corresponding to the signature $(5, 0; (1, 5), (1, 5), (3, 5))$. We may take C^* to be a chain of length 2, as shown in Figure 3.2.

When $p = 7$, $g(p) = 3$. Up to conjugacy and powers there are two elements to consider, corresponding to the signatures $(7, 0; (1, 7), (1, 7), (5, 7))$ and $(7, 0; (1, 7), (2, 7), (4, 7))$. In each case we may take C^* to be a chain of length 3, as shown in Figure 3.2.

When $p = 11$, $g(p) = 5$. Up to conjugacy and powers there are two elements to consider, corresponding to the signatures $(11, 0; (1, 11), (1, 11), (9, 11))$ and $(7, 0; (1, 11), (2, 11), (8, 11))$. In each case we may take C^* to be a chain of length 3; the pictures are similar to those for $p = 7$. □

Lemma 3.4. *Let p be either an odd prime or 4. Let $g \geq 3$ and let $f \in \mathrm{Mod}(S_g)$ be a reducible element of order p. Then there exists a set of curves C^* in S_g such that*

- *C^* is a subset of a maximal non-separating chain of curves in S_g,*
- *$|C^*| \geq g/4$,*
- *every orbit of curves under the action of $\langle f \rangle$ that contains a curve in C^* contains at least two curves in C^*, and*
- *each subset of curves in C^* that lies in the same orbit under $\langle f \rangle$ forms either a chain of length at least 2 or else consists of disjoint curves.*

Proof. Let p, g, and f be as in the statement. We first observe that if two sets of curves Y_1 and Y_2 are each subsets of a maximal non-separating chain of curves in S_g and the Y_i lie in disjoint subsurfaces of S_g that are distinct in homology, then $Y_1 \cup Y_2$ is again a

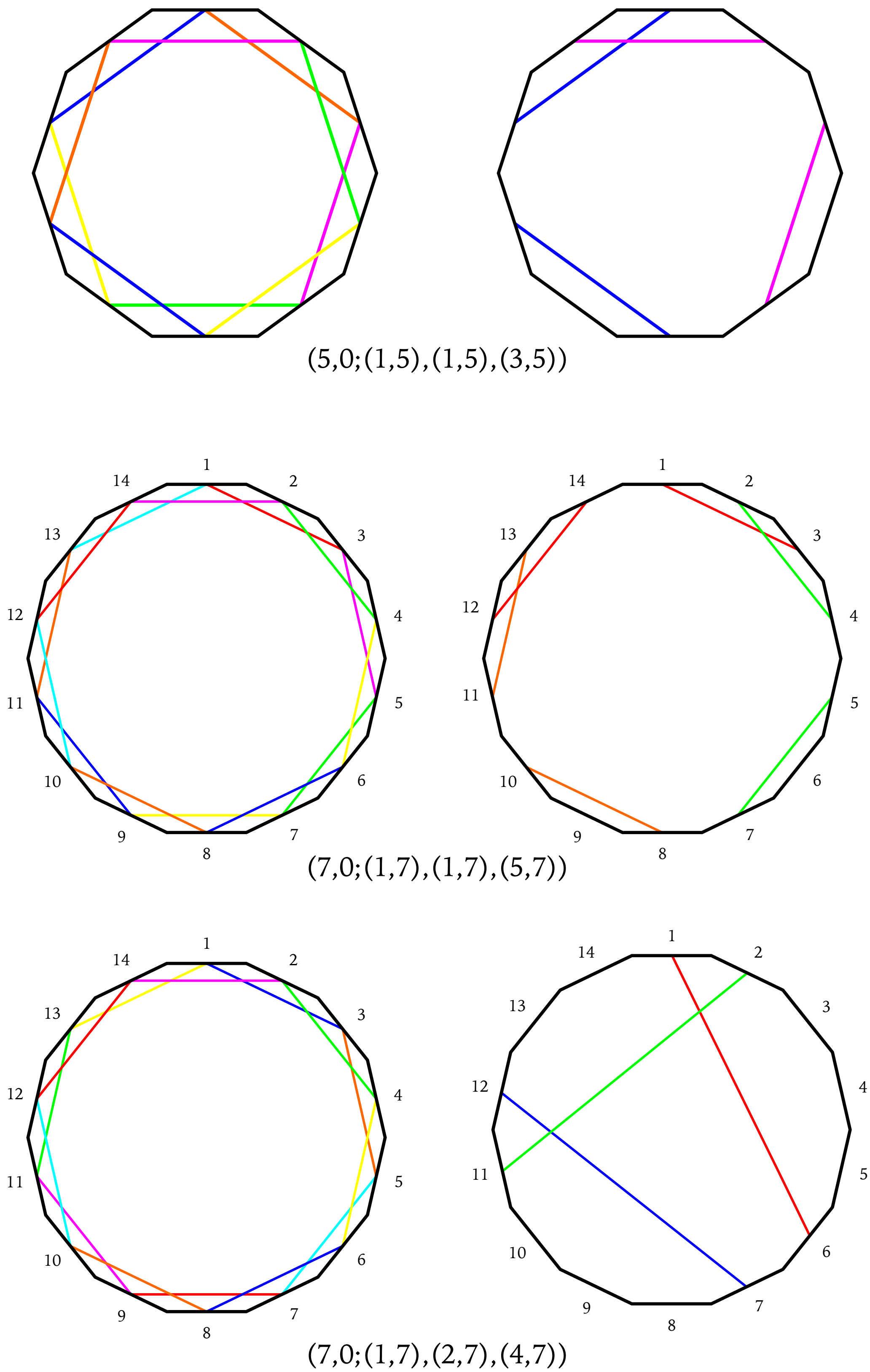

FIGURE 3.2. The collections of curves C^* for the irreducible periodic mapping classes of orders 5 and 7.

subset of a chain of curves in S_g. We will use this fact freely in forming our collection of curves C^*.

We first treat the case when p is an odd prime. Since f is reducible, the quotient orbifold $Q = S_g/\langle f\rangle$ has either $g_0 > 0$ or $\ell > 3$ or both, by a result of Gilman [5]. (Here g_0 is the genus of Q and ℓ is the number of orbifold points of Q.) If $g_0 > 0$, we may take preimages under $\langle f\rangle$ of g_0 disjoint separating curves in Q, each of which cuts off a single handle from Q. In the preimage, the curves corresponding to a single separating curve in Q cut off an orbit of p handles, distinct in homology. We may take a chain of two non-separating curves in each handle and produce all together $2p$ disjoint curves, which together form a subset of a chain of curves in S_g. These free orbits of handles, then, have a curve-to-genus ratio of 2/1. Since we are only showing that we can form C^* so that the curves-to-genus ratio is 1/4, orbits of this type can only help the ratio.

We have therefore reduced to the case of elements f such that $g_0 = 0$. Since f is assumed reducible, again by a result of Gilman [5], f has a reduction system of disjoint essential curves $\mathcal{C}$ that are pairwise non-isotopic, which we may take to be maximal. Note that every orbit of curves in $\mathcal{C}$ under $\langle f\rangle$ has size 1 or p, since p is prime. Let $S_g(\mathcal{C})$ denote the (possibly disconnected) surface obtained from S_g by cutting along $\mathcal{C}$ and capping the resulting boundary components with disks containing a single marked point. Corresponding to f there is an action f' on $S_g(\mathcal{C})$.

We now consider the connected components R_i of $S_g(\mathcal{C})$. We first claim that f' induces the identity permutation on the R_i. Otherwise there would be an orbit of R_i of size p. If these R_i had positive genus, this would contradict the assumption of $g_0 = 0$. If they instead were spheres, they would each have at least 3 marked points in order for the curves of $\mathcal{C}$ to be essential and non-isotopic, and this would again contradict the assumption that $g_0 = 0$. Therefore each R_i is mapped to itself by f'.

We now analyze what the components R_i may be and how f' may act upon them. If R_i is a sphere, then f' acts on R_i by a rotation of $2m\pi/p$, with $(m, p) = 1$. The action of f' on R_i has exactly two branch points, each of order p. If instead R_i is not a sphere, f' restricts to an irreducible self-map of order p, and we classified these in Lemma 3.2. In particular, each non-sphere R_i has genus $\frac{p-1}{2}$ and has exactly 3 branch points, each of order p.

We are now prepared to recover the action of f on S_g by reglueing annuli at pairs of marked points. For any R_i that is a sphere, it cannot only have marked points at one or both of its branch points, since the curves of $\mathcal{C}$ are essential and non-isotopic. Therefore each sphere R_i has some $r_i > 0$ orbits of p marked points, each having 0 local rotation number. Each of the p marked points in each orbit is paired with p other marked points that lie in some other R_j. (We have that $i \neq j$ by the assumption that $g_0 = 0$.) For each such orbit of annuli added, we may add to C^* a collection of $p-1$ disjoint non-separating curves (being sure not to double count). The sphere R_i may additionally have marked points at one or both of its branch points; each can either connect to a marked point on a different R_j, or else they may together form a pair. From these orbits we do not take any curves to add to C^*.

In the other case, where R_i is a surface with positive genus, R_i may have marked points at its branch points under the action of f', or at non-branch points, or at both. The surface R_i has some $r_i \geq 0$ orbits of p marked points (possibly none), each having 0 local rotation number. Each of the p marked points in each orbit is paired with p other marked points that lie in some other R_j. (We have that $i \neq j$ by the assumption that $g_0 = 0$.) For each such orbit of annuli, we may add to C^* a collection of $p-1$ disjoint non-separating curves (again, being sure not to double count). The surface R_i may additionally have marked points at its branch points; each can either connect to a marked point on a different R_j, or two of these marked points may together form a pair. In fact, this last possibility does not arise, since then $c_i + c_j = 0 \pmod p$ but we have that $c_1 + c_2 + c_3 = 0 \pmod p$ and that each c_i is nonzero $(\mathrm{mod}\ p)$. In any case, no further contribution to C^* is made for these orbits. Also, we have by Lemma 3.3 that R_i supports $p/4 - 1$ curves to contribute to C^* whenever $g \geq 13$, and either 2 or 3 curves for smaller values of p.

We are now prepared to compute bounds for each of g and $|C^*|$ in terms of the data described so far. Let a be the number of R_i that are spheres and let b be the number of R_i that are surfaces with positive genus. Then the number of branch points of the R_i totals $2a + 3b$. Therefore the maximal number of marked branch points is the same, and the number of fixed annuli is at most $\frac{2a+3b}{2}$. Let $k = \frac{1}{2}\sum_i r_i$ be the number of orbits of annuli of size p. Since $(a + b - 1)$ of these annuli are required so that the resulting surface is connected—and therefore do not contribute to the genus—we have the following bound for g:

$$g \leq a \cdot 0 + b \cdot \frac{p-1}{2} + kp + \frac{2a+3b}{2} - (a+b-1) = \frac{bp}{2} + kp + 1$$

Note that since $g \geq 3$ by assumption, at least one of b or k is positive.

Let c be equal to the constant guaranteed by Lemma 3.3: $p/4 - 1$ when $g \geq 13$ and 2, 2, 3, or 3 when g is 3, 5, 7, or 11, respectively. As we have a contribution of c curves to C^* for each R_i of positive genus and a contribution of $p - 1$ curves from each of k free orbits of annuli (with orbit of size p), we have the following equality for $|C^*|$:

$$|C^*| = bc + k(p-1)$$

We now apply the following three facts: (1) at least one of b or k is positive; (2) in general, $\frac{q+r}{s+t} \geq \min\{\frac{q}{s}, \frac{r}{t}\}$ for $q, r, s, t > 0$; (3) the individual inequalities

$$\frac{bc}{\frac{bp}{2} + 1} \geq \frac{1}{4} \qquad \text{and} \qquad \frac{k(p-1)}{kp+1} \geq \frac{1}{4}$$

hold for each $p \geq 3$ and its corresponding value of c whenever b is positive and whenever k is positive, respectively.

Applying these facts yields the desired result:

$$\frac{|C^*|}{g} \geq \frac{bc + k(p-1)}{\frac{bp}{2} + kp + 2} \geq \frac{1}{4}.$$

The case when $p = 4$ follows the same outline. □

Proof of the main theorem. With all of our preliminaries in hand, we are prepared to prove Theorem 1.4.

Proof of Theorem 1.4. Let $g \geq 3$ and let $H_g = \{a_1, \ldots, a_{2g}, b\}$ be a set of simple closed non-separating curves corresponding to the Humphries generating set for $\mathrm{Mod}(S_g)$. Let f be a periodic element of $\mathrm{Mod}(S_g)$ with $|f| \geq 3$. By passing to a power, we may assume without loss of generality that $|f|$ is either 4 or an odd prime. By Lemma 3.1, a Dehn twist about a non-separating curve may be written as a product in at most 12 distinct conjugates of f. Take 12 conjugates of f in which T_b is a product and let them be the start to our generating set for $\mathrm{Mod}(S_g)$.

By Lemmas 3.3 and 3.4, f acts on C^*, a subset of a maximal non-separating chain of curves in S_g, such that $|C^*| \geq g/4$ and every orbit of curves under the action of $\langle f \rangle$ that contains a curve in C^* contains at least two curves in C^*. The C^* curves that lie in a given orbit either form a chain of length at least 2, or they are all disjoint. We consider two cases, depending on whether at least half of the curves of C^* lie in orbits of the former type, or of the latter type. In either case, let this collection of curves be called C'. We have $|C'| \geq g/8$. We treat the two cases in turn, in similar fashion.

First case: chains. When the curves of C' all lie in chains of length at least 2, we form at most $16 = \frac{2g}{g/8}$ subsets of $H_g - \{b\}$, $C'_1, \ldots, C'_{16}$, that are each of the topological type of C' and that together cover $H_g - \{b\}$ except for "gaps" of size 1. See Figure 3.3. Note that fewer than 16 subsets may be required, and that it does not matter how the "overflow" curves for the last C'_i are chosen. We may then take at most 16 conjugates of f, $\{f_1, \ldots, f_{16}\}$, so that each individually acts on the corresponding C'_i in the way that f acts on C'. In particular, each curve in each C'_i is in the same orbit as the other curves belonging to the same chain in C'_i under the action of the cyclic group $\langle f_i \rangle$.

Similarly, we form $C'_{17}, \ldots, C'_{32}$ and $C'_{33}, \ldots, C'_{48}$ that are shifted one and two curves down the chain $H_g - \{b\}$ from the corresponding curves in $C'_1, \ldots, C'_{16}$. Note that it again does not matter where the "overflow" curves are chosen, except that we ensure that b is among them. Again, see Figure 3.3. We may take 32 conjugates of f, $\{f_{17}, \ldots, f_{48}\}$, so that each acts on C'_i in the way that f acts on C', for $17 \leq i \leq 48$.

We now argue that all of the curves in H_g are in the same orbit under the subgroup generated by the 48 f_i. All but the "gap" curves are put into orbits by $\langle f_1, \ldots, f_{16} \rangle$. Each gap curve is put into the same orbit as the curve immediately "prior" to it by $\langle f_{17}, \ldots, f_{32} \rangle$ and also into the same orbit as the curve immediately "after" it by $\langle f_{33}, \ldots, f_{48} \rangle$. Thus all of the original orbits are collapsed into a single orbit "through" the gap curves. By construction we also have that b is in this orbit.

Therefore the curves in H_g are all in the same orbit under the action of a subgroup generated by 48 conjugates of f. With the additional 12 conjugates of f we have a Dehn twist about b in the subgroup, and so also the Dehn twists about all curves in H_g, and therefore the subgroup so generated is equal to $\mathrm{Mod}(S_g)$. Thus $\mathrm{Mod}(S_g)$ is generated by 60 conjugates of f, as required.

Second case: disjoint curves. When the curves of C' are all disjoint, we form at most $8 = \frac{g}{g/8}$ subsets of $H_g - \{b\}$, $C'_1, \ldots, C'_8$, that are each of the topological type of C' and that together cover the a_i curves in H_g with odd indices. Call these subsets C'_{odd}. See Figure 3.4. Note that fewer than 8 subsets may be required, and that it does not matter how the "overflow" curves for the last C'_i are chosen. We may then take at most 8 conjugates of f, $\{f_1, \ldots, f_8\}$, so that each individually acts on the corresponding C'_i in the way that f acts on C'. Similarly, we form sets of curves $C'_9, \ldots, C'_{16}$ that are each of the topological type of C' and that together cover the a_i curves in H_g with even indices; we may do this by shifting all of the C'_{odd} curves down the $H_g - \{b\}$ chain by one curve. Call these subsets C'_{even}. We take at most 8 conjugates of f, $\{f_9, \ldots, f_{16}\}$, so that each individually acts on the corresponding C'_i in the way that f acts on C'.

We now take $C'_{17}, \ldots, C'_{24}$ and $C'_{25}, \ldots, C'_{32}$ that are shifted two and three curves down the chain $H_g - \{b\}$ from the C'_{odd} curves. (So two down from C'_{odd} and then two down from C'_{even}.) Note that fewer than 16 subsets may be required, and that it again does not matter where the "overflow" curves are chosen, except that we ensure that b is among them, and also that an "overflow" curve from the shifted "odd" chain is one of the a_i curves with an even index. Again, see Figure 3.4. We take at most 16 conjugates of f, $\{f_{17}, \ldots, f_{32}\}$, so that each acts on C'_i in the way that f acts on C', for $17 \leq i \leq 32$.

We now argue that all of the curves in H_g are in the same orbit under the subgroup generated by the 32 f_i. All of the a_i curves with odd index are in the same orbit, since all of their orbits under $\langle f_1, \ldots, f_8 \rangle$ are collapsed by the "shifted" conjugates. Similarly, all of the a_i curves with even index are in the same orbit under $\langle f_1, \ldots, f_{32} \rangle$. By construction, there exist an even a_i curve and an odd a_i curve that lie in the same orbit under $\langle f_1, \ldots, f_{32} \rangle$, and also b is in the same orbit as some a_i curve. Therefore the curves in H_g are all in the same orbit under the action of a subgroup generated by 32 conjugates of f.

With an additional 12 conjugates of f we have a Dehn twist about b in the subgroup, and so also the Dehn twists about all curves in H_g, and therefore the subgroup so generated is equal to $\mathrm{Mod}(S_g)$. Thus $\mathrm{Mod}(S_g)$ is generated by 44 conjugates of f, as required. $\square$

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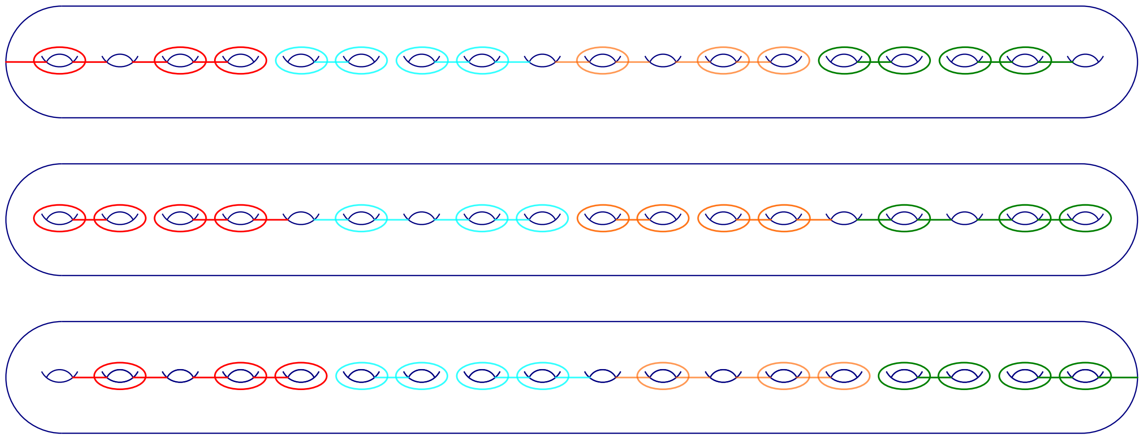

FIGURE 3.3. A schematic for the collections of curves $C'_1, \ldots, C'_{16}$, $C'_{17}, \ldots, C'_{32}$, and $C'_{33}, \ldots, C'_{48}$. Within a single surface, curves in different C'_i are in different colors, and within a collection C'_i the curves in the same orbit under the cyclic group $\langle f_i \rangle$ are in the same chain. Note that for the case illustrated, only four collections of curves are needed in each of the three groupings, and also that "overflow" curves are not depicted.

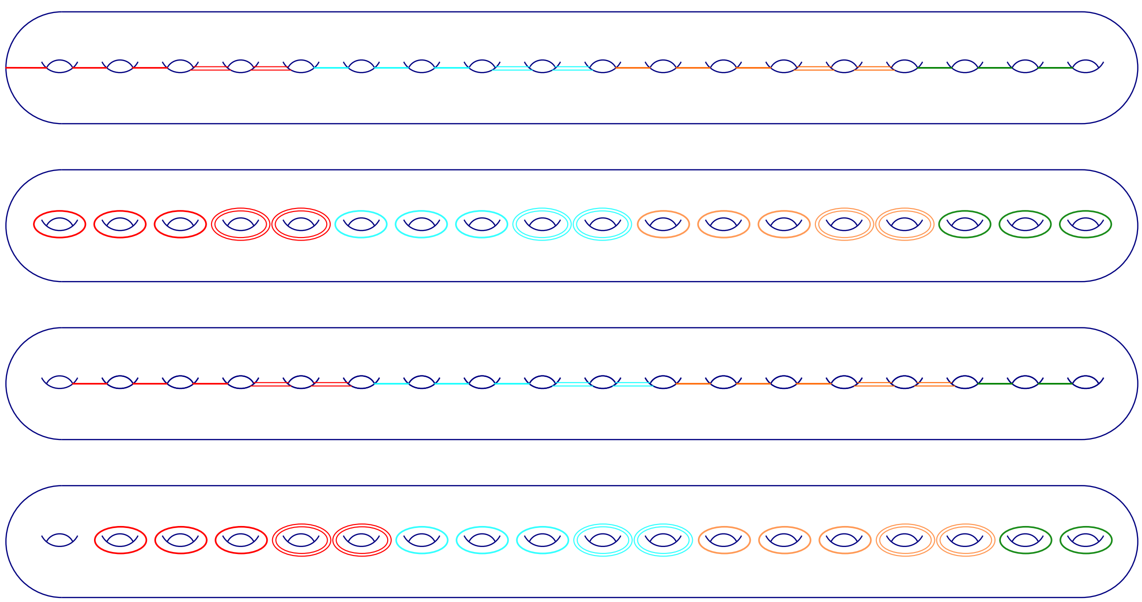

FIGURE 3.4. A schematic for the collections of curves C'_{odd}, C'_{even}, and the shifted versions of each. Within a single surface, curves in different C'_i are in different colors, and within a collection C'_i the curves in the same orbit under the cyclic group $\langle f_i \rangle$ have matching markings. Note that for the case illustrated, only four C'_i are needed in each of the four groupings, and also that "overflow" curves are not depicted.